CONSTRUCTION AND ANALYSIS OF CONFOUNDED DESIGNS $q \times 3^n$ THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

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1. INTRODUCTION

Recently several methods of construction of asymmetrical factorial designs have been evolved. Kishen and Srivastava (1959) obtained the designs $q \times S^n$ through hyper surfaces defined in finite geometries. Das (1960) linked the construction of these designs with the fractional replications of some corresponding symmetrical factorial designs. Both Kishen and Das advanced further methods of construction of balanced designs $q \times 2^2$ in $2q$ plot blocks through balanced incomplete block (B.I.B.) designs. Kishen (1960), Kishen and Tyagi (1961) presented further methods of construction of $q \times 2^2$ and $q \times 3^2$ designs through P.B I.B. designs following Shah's (1960) technique. In all these works attempts have been made to give balanced designs, which in many cases need large number of replications, and hence large amount of resources

As pointed out by Sardana and Das (1965), it is desirable to give designs involving smallest feasible number of replications providing mutually independent estimates of all the effects so as to bring about economy in the use of resources. It can be even said that the design need not necessarily be a resolvabe one and also equi-replicated. Keeping these points in view, Sreenath (1965) gave methods of construction of designs $q \times 2^n$ in $q \times 2^p$ plot blocks through P.B.I.B. designs with b and b-half replications, where b is the number of blocks in the P.B.I.B. design used. Sreenath has also given methods of construction of designs $2q \times 2^n$ in $2q \times 2^p$ plot blocks involving only two replications. This was followed by further methods of construction of designs $K^2 \times 2^n$ in $K^2 \times 2^p$ plot blocks involving $2k$ replications by Sardana and Sreenath (1965) and designs $(2q+1) \times 2^n$ in $(2q+2) \times 2^p$ plot blocks by Sreenath and Sardana

(1967). In the present paper a method of construction of designs $q\times 3^n$ in $q\times 3^p$ plot blocks through B.I.B. designs has been presented alongwith the method of analysis, keeping the above points in view.

2. Design $q\times 3^2$ in $3q$ Plot Blocks with b Replications

We shall consider, in the first instance, the case of the design $q\times 3^2$ in $3q$ plot blocks. The generalisation to the case of design $q\times 3^n$ in $q\times 3^p$ plot blocks, which can be done following Sreenath's (1965) technique, is indicated in section 5.

2·1. Construction

Let the factors and their levels be denoted as under :

Factors	*Levels*
A	$a_0, a_1, a_2, \ldots, a_{q-1}$
B	b_0, b_1, b_2
C	c_0, c_1, c_2.

For the construction of the design we shall make use of a set of three B.I.B. designs as described below.

Let α, β and γ denote respectively the three sets of treatment combinations, say,

$$(b_0c_0,\ b_1c_1,\ b_2c_2)\ ;\ (b_0c_2,\ b_1c_0,\ b_2c_1)\ ;$$

and
$$(b_0c_1,\ b_1c_2,\ b_2c_0)$$

of the factors B and C. The interaction BC^2 with 2 *d.f.* can be considered to be the comparison between α, β and γ. We have now to obtain a set of three B.I.B. designs with the q levels of the factor A as the treatments and the following parameters:

B.I.B. Design	*Parameters*
D_1	$v=q,\ k_1,\ r_1,\ b,\ \lambda_1$
D_2	$v=q,\ k_2,\ r_2,\ b,\ \lambda_2$
D_3	$v=q,\ k_3,\ r_3,\ b,\ \lambda_3$

where $k_1+k_2+k_3=q$ and the B.I.B. designs are such that the sum of the three $q\times b$ incidence matrices of these designs is a $q\times b$ matrix with each of its elements as unity.

Replacing the elements 1 in the incidence matrices of D_1, D_2 and D_3 by α, β and γ respectively, let us sum these incidence matrices and denote this sum by $M_{\alpha\beta\gamma}$. By using the method that follows, each of the b columns of $M_{\alpha\beta\gamma}$ will be used to generate a block of the design $q \times 3^2$. Let us take, say, the m-th column of $M_{\alpha\beta\gamma}$. By associating the i-th row element of this column with the level a_{i-1} of the factor A for $i = 1, 2, \dots, q$ and then replacing

(*i*) $a_i\alpha$ by the three combinations $(a_ib_0c_0,\ a_ib_1c_1,\ a_ib_2c_2)$;

(*ii*) $a_j\beta$ by $(a_jb_0c_2,\ a_jb_1c_0,\ a_jb_2c_1)$; and

(*iii*) $a_k\gamma$ by $(a_kb_1c_2,\ a_kb_0c_1,\ a_kb_2c_0)$

a block of size $3q$ of the design $q \times 3^2$ is generated.

Similarly using all the b columns of $M_{\alpha\beta\gamma}$ we generate b blocks of the design $q \times 3^2$, each of size $3q$ plots. Now by changing the order (α, β, γ) for replacement of the elements 1 in the incidence matrices of D_1, D_2 and D_3 to (β, γ, α) and (γ, α, β) we obtain the matrices $M_{\beta\gamma\alpha}$ and $M_{\gamma\alpha\beta}$ respectively and then generate, as above, b blocks each of size $3q$ plots of the design from b columns of each of $M_{\beta\gamma\alpha}$ and $M_{\gamma\alpha\beta}$. Thus the asymmetrical factorial design $q \times 3^2$ can be obtained in $3b$ blocks each of size $3q$. The number of replications in the design is evidently b.

As an example we construct the design 7×3^2 in 21 plot blocks. Let the incidence matrices of the three B.I.B. designs, with the 7 levels of the factor A as treatments, that we make use of in the construction, be :

B.I.B. design	D_1 ($v=b=7,\ k_1=r_1=3,\ \lambda_1=1$)	D_2 ($v=b=7,\ k_2=r_2=1,\ \lambda_2=0$)	D_3 ($v=b=7,\ k_3=r_3=3,\ \lambda_3=1$)
Incidence Matrix	$\begin{bmatrix} 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0\\ 0&0&0&1&0&1&1 \end{bmatrix}$	$\begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&1&1&0&1&0\\ 0&0&0&1&1&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0\\ 0&1&1&0&1&0&0 \end{bmatrix}$

Then we have

$$M_{\alpha\beta\gamma}=\begin{bmatrix} \alpha & \beta & \gamma & \gamma & \alpha & \gamma & \alpha \\ \alpha & \alpha & \beta & \gamma & \gamma & \gamma & \alpha \\ \gamma & \alpha & \alpha & \beta & \gamma & \gamma & \alpha \\ \alpha & \gamma & \alpha & \alpha & \beta & \gamma & \gamma \\ \gamma & \alpha & \gamma & \alpha & \alpha & \beta & \gamma \\ \gamma & \gamma & \alpha & \gamma & \alpha & \alpha & \beta \\ \beta & \gamma & \gamma & \alpha & \gamma & \alpha & \alpha \end{bmatrix} \quad M_{\beta\gamma\alpha}=\begin{bmatrix} \beta & \gamma & \alpha & \alpha & \beta & \alpha & \beta \\ \beta & \beta & \gamma & \alpha & \alpha & \beta & \alpha \\ \alpha & \beta & \beta & \gamma & \alpha & \alpha & \beta \\ \beta & \alpha & \beta & \beta & \gamma & \alpha & \alpha \\ \alpha & \beta & \alpha & \beta & \beta & \gamma & \alpha \\ \alpha & \alpha & \beta & \alpha & \beta & \beta & \gamma \\ \gamma & \alpha & \alpha & \beta & \alpha & \beta & \beta \end{bmatrix} \quad M_{\gamma\alpha\beta}=\begin{bmatrix} \gamma & \alpha & \beta & \beta & \gamma & \beta & \gamma \\ \gamma & \gamma & \alpha & \beta & \beta & \gamma & \beta \\ \beta & \gamma & \gamma & \alpha & \beta & \beta & \gamma \\ \gamma & \beta & \gamma & \gamma & \alpha & \beta & \beta \\ \beta & \gamma & \beta & \gamma & \gamma & \alpha & \beta \\ \beta & \beta & \gamma & \beta & \gamma & \gamma & \alpha \\ \alpha & \beta & \beta & \gamma & \beta & \gamma & \gamma \end{bmatrix}$$

Thus the design 7×3^2 in 21 plot blocks will be :

Level of A	Blocks from $M_{\alpha\beta\gamma}$							Blocks from $M_{\beta\gamma\alpha}$							Blocks from $M_{\gamma\alpha\beta}$						
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
a_0	α	β	γ	γ	α	γ	α	β	γ	α	α	β	α	β	γ	α	β	β	γ	β	γ
a_1	α	α	β	γ	γ	α	γ	β	β	γ	α	α	β	α	γ	γ	α	β	β	γ	β
a_2	γ	α	α	β	γ	γ	α	α	β	β	γ	α	α	β	β	γ	γ	α	β	β	γ
a_3	α	γ	α	α	β	γ	γ	β	α	β	β	γ	α	α	γ	β	γ	γ	α	β	β
a_4	γ	α	γ	α	α	β	γ	α	β	α	β	β	γ	α	β	γ	β	γ	γ	α	β
a_5	γ	γ	α	γ	α	α	β	α	α	β	α	β	β	γ	β	β	γ	β	γ	γ	α
a_6	β	γ	γ	α	γ	α	α	γ	α	α	β	α	β	β	α	β	β	γ	β	γ	γ

It can be seen that the number of replications in the design is 7 with the blocks (1, 8, 15) forming one replication and other replications are formed by the blocks (2, 9, 16) ; (3, 10, 17) ; ... ; (7, 14, 21).

2·2. Analysis

In the design, constructed as above, only the interaction BC^2 with 2 *d.f.* and $A(BC^2)$ with $2(q-1)$ *d.f.* are affected by block differences. The interaction $A(BC^2)$ with $2(q-1)$ *d.f.* can be considered to be made up of two components *viz.* ;

$A(BC^2)_L$ and $A(BC^2)_Q$ each with $(q-1)$ *d.f.* where $(BC^2)_L$ and $(BC^2)_Q$ correspond to the two contrasts $(\alpha-\gamma)$ and $(\alpha-2\beta+\gamma)$ of the interaction BC^2 respectively. The following scheme of constants has been used in writing the normal equations and the estimates corresponding to the effects of the affected interactions BC^2 and $A(BC^2)$.

Interaction	*D.F.*	*Effects*
BC^2	2	t_0, t_1, t_2
$A(BC^2)_L$	$(q-1)$	$u_0, u_1, \ldots, u_{q-1}$
$A(BC^2)_Q$	$(q-1)$	$v_0, v_1, \ldots, v_{q-1}$.

The normal equations for estimating the parameters after eliminating the block effects under the restrictions $\Sigma_i t_i=0$, $\Sigma_i u_i=0$. and $\Sigma_i v_i=0$ come out as

$$P_i=\left[3bq-\frac{9b\{(k_1-k_2)^2+(k_2-k_3)^2+(k_3-k_1)^2\}}{6q}\right]t_1 \quad \text{for } i=0, 1, 2 \qquad \ldots(1)$$

$$Q_i=\left[6b-\frac{27(b-\lambda_1-\lambda_2-\lambda_3)}{3q}\right]u_i-f(t) \quad \text{for } i=0, 1, \ldots, (q-1) \qquad \ldots(2)$$

$$R_i=\left[18b-\frac{81(b-\lambda_1-\lambda_2-\lambda_3)}{3q}\right]v_i-f'(t) \quad \text{for } i=0, 1, \ldots, (q-1) \qquad \ldots(3)$$

where $f(t)=(3/b)\ (r_1^2+r_2^2+r_3^2-r_1r_2-r_2r_3-r_3r_1)\ \{t_0-t_2\}$

$f'(t)=(3/b)\ (r_1^2+r_2^2+r_3^2-r_1r_2-r_2r_3-r_3r_1)\ \{t_0-2t_1+t_2\}$

P_i, Q_i and R_i are respectively $(BC^2)_i$, $[A(BC^2)_L]_i$ and $[A(BC^2)_Q]_i$ after adjusting for block effects, *i.e.*,

$(BC^2)_i$=Sum of observations from the treatment combinations involving the combinations in α, β and γ respectively for $i=0, 1, 2$

$[A(BC^2)_L]_i=a_i(BC^2)_0-a_i(BC^2)_2$ for $i=0, 1, \ldots, (q-1)$

$[A(BC^2)_Q]_i=a_i(BC^2)_0-2a_i(BC^2)_1+a_i(BC^2)_2$ for $i=0, 1,\ldots, (q-1)$

$a_i(BC^2)_j$=Sum of observations from the treatment combinations with the i-th level of A and involving the combinations in α, β and γ respectively for $j=0$, 1 and 2.

Adjustment for block effects in $(BC^2)_i=-(1/3q)\ \Sigma_j\ \delta_{ij}\ B_j$

where δ_{ij}=number of treatment combinations in the j-th block contributing to the sum $(BC^2)_i$

B_j=j-th block total.

Adjustment for block effects in $[A(BC^2)_L]_i = -(1/q)\, \Sigma_j\, \epsilon_{ij}\, B_j$ where $\epsilon_{ij} = +1$ or 0 -1 according as the i-th level of A occurs in the j-th block with the combinations in α or β or γ.

Adjustment for block effects in $[A(BC^2)_Q]_i = -(1/q)\, \Sigma_j\, \eta_{ij}\, B_j$ where $\eta_{ij} = +1$ or -2 or $+1$ according as the i-th level of A occurs in the j-th block with the combinations in α or β or γ.

Thus we have here

$$P_0 = (BC^2)_0 - (1/q)(k_1T_1 + k_2T_3 + k_3T_2)$$
$$P_1 = (BC^2)_1 - (1/q)(k_1T_2 + k_2T_1 + k_3T_3)$$
$$P_2 = (BC^2)_2 - (1/q)(k_1T_3 + k_2T_2 + k_3T_1)$$

where $T_1 = \sum_{i=1}^{b} B_i$; $T_2 = \sum_{i=b+1}^{2b} B_i$; $T_3 = \sum_{i=2b+1}^{3b} B_i$. ...etc.

It may be noted here that the functions $f(t)$ and $f'(t)$ occurring in the right hand side of the normal equations for interactions $A(BC^2)_L$ and $A(BC^2)_Q$ do not appear in the estimates of the contrasts of interaction effects u_i's and v_i's obtained from these normal equations. Since our interest is in the estimates of the contrasts of the effects and not in the effects as such, the presence of these functions in the normal equations can be ignored while obtaining the estimates of the interaction effects. We thus have the estimates of the interaction effects as

$$\hat{t}_i = \frac{6q}{18bq^2 - 9b\{(k_1-k_2)^2 + (k_2-k_3)^2 + (k_3-k_1)^2\}}\, P_i \quad \text{for } i=0,\ 1,\ 2 \qquad \text{...(4)}$$

$$\hat{u}_i = \frac{3q}{18bq - 27(b-\lambda_1-\lambda_2-\lambda_3)}\, Q_i \text{ for } i=0,\ 1, ..., (q-1) \qquad \text{...(5)}$$

$$\hat{v}_i = \frac{3q}{54bq - 81(b-\lambda_1-\lambda_2-\lambda_3)}\, R_i \text{ for } i=0,\ 1, ..., (q-1) \qquad \text{..(6)}$$

The sum of squares (S.S.) due to BC^2, $A(BC^2)_L$ and $A(BC^2)_Q$ can be obtained as

$$\sum_{i=0}^{2} \hat{t}_i P_i \ ; \quad \sum_{i=0}^{(q-1)} \hat{u}_i Q_i - (\Sigma_i Q_i)(\Sigma_i \hat{u}_i)/q \ ;$$

and

$$\sum_{i=0}^{(q-1)} \hat{v}_i R_i - (\Sigma_i R_i)(\Sigma_i \hat{v}_i)/q \quad \text{respectively.}$$

The relative loss of information on (*i*) BC^2 with 2 d.f. will be $(1/2q^2)\{(k_1-k_2)^2+(k_2-k_3)^2+(k_3-k_1)^2\}$; (*ii*) $A(BC^2)$ with $2(q-1)$ d.f. will be $(3/2bq)(b-\lambda_1-\lambda_2-\lambda_3)$. The total relative loss of information in the design will thus be 2(= number of blocks per replication *minus* one). It may be noted that this design is a resolvable design.

As an example let us consider the design 7×3^2 in 21 plot blocks obtained in section 2·1. We find that the normal equations, in this case, for estimating the parameters after eliminating the block effects under the restrictions $\Sigma_i\, t_i=0$; $\Sigma_i u_i=0$; $\Sigma_i v_i=0$ will come out as

$$P_i=135t_i \text{ for } i=0, 1 \text{ and } 2 \qquad \ldots(1a)$$

$$Q_i=(42-45/7)u_i-(12/7)(t_0-t_2) \quad \text{for} \quad i=0, 1,\ldots, 6 \qquad \ldots(2a)$$

$$R_i=(126-135/7)v_i-(12/7)(t_o-2t_1+t_2) \text{ for } i=0, 1,\ldots, 6 \qquad \ldots(3a)$$

where P_0=(Sum of observations from the treatment combinations in $a_0\alpha$, $a_1\alpha$, $a_2\alpha$, $a_3\alpha$, $a_4\alpha$, $a_5\alpha$ and $a_6\alpha$)$-(3/7)\sum_{i=1}^{14} B_i-(1/7)\sum_{i=15}^{21} B_i$

Q_0=(Sum of observations from the treatment combinations in $a_0\alpha$)—(Sum of observations from the treatment combinations in $a_0\gamma$)$-(1/7)(B_1-B_3-B_4+B_5-B_6+B_7-B_9+B_{10}+B_{11}+B_{13}-B_{15}+B_{16}-B_{19}-B_{21})$

R_0=(Sum of observations from the treatment combinations in $a_0\alpha$ and $a_0\gamma$)—2(Sum of observations from the treatment combinations in $a_0\beta$)$-(1/7)(B_1-2B_2+B_3+B_4+B_5+B_6+B_7-2B_8+B_9+B_{10}+B_{11}-2B_{12}+B_{13}-2B_{14}+B_{15}+B_{16}-2B_{17}-2B_{18}+B_{19}-2B_{20}+B_{21})$...etc.

Thus we have

$$\hat{t}_i=(1/135)P_i \quad \text{for} \quad i=0, 1, 2 \qquad \ldots(4a)$$

$$\hat{u}_i=(7/249)Q_i \quad \text{for} \quad i=0, 1,\ldots, 6 \qquad \ldots(5a)$$

$$\hat{v}_i=(7/747)R_i \quad \text{for} \quad i=0, 1,\ldots, 6 \qquad \ldots(6a)$$

S.S. due to BC^2 with 2 d.f.$=(1/135)\ \Sigma\ P^2_i$

S.S. due to $A(BC^2)_L$ with 6 d.f.$=(7/249)\left\{\Sigma\ Q^2_i-(\Sigma\, Q_i)^2/7\right\}$

S.S. due to $A(BC^2)_Q$ with 6 d.f.$=(7/747)\left\{\Sigma\ R^2_i-(\Sigma R_i)^2/7\right\}$

The relative loss of information on BC^2 with 2 d.f. is 4/49 and on $A(BC^2)$ with 12 d.f. is 15/98.

3. Design $q\times 3^2$ with $2b.\ (1/3)$ Replications

We shall, in this section, consider a method of construction by which it will be possible to reduce the requirements of the experimental material by 1/3.

3·1. Construction

The method of construction is on similar lines indicated in section 2·1. The first set of b blocks is generated from the b columns of $M_{\alpha\beta\gamma}$ The second set of b blocks is generated from the b columns of $M_{\gamma\beta\alpha}$, where $M_{\gamma\beta\alpha}$ is obtained similarly as $M_{\alpha\beta\gamma}$ excepting that the order (α, β, γ) for replacement of elements 1 in the incidence matrices of D_1, D_2 and D_3 is changed to (γ, β, α). Thus in all $2b$ blocks each of size $3q$ of the design $q\times 3^2$ are obtained. Each of the blocks is a 1/3 replication and there are therefore $2b.(1/3)$ replications in the design.

3·2. Analysis

In this design also only the interactions BC^2 with 2 d.f. and $A(BC^2)$ with $2(q-1)$ d.f. are affected by block differences and are estimated independently. Using the same scheme of constants in writing the estimates and same notations as in section 2·2, the normal equations for estimating the parameters, after eliminating the block effects, under the restriction $\Sigma\, u_i=0$ and $\Sigma\, v_i=0$ come out as

$$P_0=\left\{3b(k_1+k_3)-\frac{9b(k^2_1+k^2_3)}{3q}\right\}t_0-\frac{9bk_2(k_1+k_3)}{3q}\,t_1-\frac{18bk_1k_3}{3q}\,t_2$$

$$P_1=\left\{6bk_2-\frac{18bk^2_2}{3q}\right\}t_1-\frac{9bk_2(k_1+k_3)}{3q}\,(t_0+t_2)$$

$$P_2=\left\{3b(k_1+k_3)-\frac{9b(k^2_1+k^2_3}{3q}\right\}t_2-\frac{9bk_2(k_1+k_3)}{3q}\,t_1-\frac{18bk_1k_3}{3q}\,t_0 \quad \ldots(7)$$

$$Q_i=\left\{6(r_1+r_3)-\frac{18(2x_1+2x_3-x_2)}{3q}\right\}u_i-\frac{9(r_1-r_3)(k_1+k_3)}{3q}(t_0-t_2)$$

$$\text{for } i=0, 1,\ldots, (q-1) \quad \ldots(8)$$

$$R_i=\{6r_1+24r_2+6r_3-(162x_2/3q)\}\, v_i-\{9(r_1-2r_2+r_3)/3q\}$$
$$\{(k_1+k_3)(t_0+t_2)-2k_2t_1\} \quad \text{for } i=0, 1,\ldots, (q-1) \quad \ldots(9)$$

where P_i, Q_i and R_i are the totals $(BC^2)_i$, $[A(BC^2)_L]_i$, and $[A(BC^2)_Q]_i$ respectively after adjusting for block effects, $x_i=r_i-\lambda_i$ for $i=1, 2, 3$.

The above equations (7), (8) and (9) can easily be solved for estimates of the effects under the restriction $(k_1+k_3)(t_0+t_2)+2k_2t_1=0$.

Thus, we have the estimates of the contrasts (t_0-t_2) and $(t_0-2t_1+t_2)$ corresponding to the interaction BC^2 as

$$\text{Estimate of } (t_0-t_2)=\frac{P_0-P_2}{3b(k_1+k_3)-\dfrac{9b(k_1-k_3)^2}{3q}}$$

$$\text{Estimate of } (t_0-2t_1+t_2)=(1/3b)\left\{\frac{P_0+P_2}{k_1+k_3}-\frac{P_2}{2k_2}\right\} \qquad \ldots(10)$$

$$\hat{u}_i=Q_i/\left\{6(r_1+r_3)-\frac{18(2x_1+2x_3-x_2)}{3q}\right\} \text{ for } i=0, 1, \ldots, (q-1) \qquad \ldots(11)$$

$$\hat{v}_i=R_i/\{6r_1+24r_2+6r_3-(162x_2/3q)\} \text{ for } i=0, 1, \ldots, (q-1) \ldots(12)$$

The S.S. due to BC^2, $A(BC^2)_L$ and $A(BC^2)_Q$ can be obtained as $\sum_i \hat{t}_iP_i$; $\Sigma\hat{u}_iQ_i-(\Sigma\hat{u}_i)(\Sigma Q_i)/q$; and $\Sigma\hat{v}_iR_i-(\Sigma\hat{v}_i)(\Sigma R_i)/q$ respectively.

The relative loss of information will be

(*i*) $1-\{3q(k_1+k_3)-3(k_1-k_3)^2\}/2q^2$ on the contrast (t_0-t_2) of BC^2

(*ii*) $1-\{9k_2(k_1+k_3)\}/2q^2$ on the constrast $(t_0-2t_1+t_2)$ of BC^2

(*iii*) $1-(3/2bq)\{q(r_1+r_3)-(2x_1+2x_3-x_2)\}$ on $A\ (BC^2)_L$

and (*iv*) $1-(1/2bq)\ \{q(r_1+r_3+4r_2)-9x_2\}$ on $A(BC^2)_Q$

The total relative loss of information will be 2(= number of blocks required per replication *minus* one).

It may be noted that this design is usually non-resolvable and also that it is not an equi-replicated design. However, when $k_1+k_3=2k_2$ the design will be equi-replicated.

As an example let us consider the following 5×3^2 design in 15 plot blocks with 20. (1/3) replications obtained through the B.I.B. designs $D_1(v=5, b=10, k_1=2, r_1=4, \lambda_1=1)$, $D_2(v=5, b=10, k_2=1, r_2=2, \lambda_2=0)$ and $D_3(v=5, b=10, k_3=2, r_3=4, \lambda_3=1)$.

Design 5×3^2 in 15 plot blocks.

Level of A	*Blocks from* $M_{\alpha\beta\gamma}$										*Blocks from* $M_{\gamma\beta\alpha}$									
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_0	α	α	α	α	β	γ	γ	γ	γ	β	γ	γ	γ	γ	β	α	α	α	α	β
a_1	α	γ	γ	β	α	α	α	β	γ	γ	γ	α	α	β	γ	γ	γ	β	α	α
a_2	γ	α	β	γ	α	γ	β	α	α	γ	α	γ	β	α	γ	α	β	γ	γ	α
a_3	β	γ	α	γ	γ	α	γ	α	β	α	β	α	γ	α	α	γ	α	γ	β	γ
a_4	γ	β	γ	α	γ	β	α	γ	α	α	α	β	α	γ	α	β	γ	α	γ	γ

The normal equations (7), (8) and (9) under the restrictions $\Sigma\, u_i = 0$, $\Sigma\, v_i = 0$, and $2\,(t_0 + t_2) + t_1 = 0$, in this case will reduce to

$P_0 = 120 t_0$; $P_1 = 60 t_1$; $P_2 = 120 t_2$; ...(7*a*)

$Q_i = 36 u_i$ for $i = 0, 1, \ldots, 5$ ...(8*a*)

$R_i = (1116/15)\, v_i$ for $i = 0, 1, \ldots, 5$ ...(9*a*)

Thus we have the estimates :

$\hat{t}_0 = P_0/120$; $\hat{t}_1 = P_1/60$; $\hat{t}_2 = P_2/120$; ...(10*a*)

$\hat{u}_i = Q_i/36$ for $i = 0, 1, \ldots, 5$ ...(11*a*)

$\hat{v}_i = 15 R_i/1116$ for $i = 0, 1, \ldots, 5$ ...(12*a*)

The S.S. due to BC^2, $A(BC^2)_L$ and $A(BC^2)_Q$ can be obtained as $\{(P^2{}_0 + P^2{}_2)/120\} + P^2{}_1/60$; $(1/36)\{\Sigma\, Q^2{}_i - (\Sigma\, Q_i)^2/5\}$; and $(15/1116)$ $\{\Sigma\, R^2{}_1 - (\Sigma\, R_1)^2/5\}$ respectively. The relative loss of information on (*i*) $(BC^2)_L$ is $-1/5$, (*ii*) $(BC^2)_Q$ is $7/25$, (*iii*) $A(BC^2)_L$ is $1/10$ and (*iv*) $A(BC^2)_Q$ is $19/50$ in this design. The total relative loss of information is $(-1/5) + (7/25) + (1/10) \times 4 + (19/50) \times 4 = 2$.

4. Design $q \times 3^2$ with b. (1/3) Replications.

It is possible to further reduce the requirements of the experimental resources to 1/3 only. This reduction can be brought in when the block sizes k_1 and k_3 are equal, *i.e.*, say $k_1 = k_3 = k$ and consequently $r_1=r_3=r$, $\lambda_1=\lambda_3=\lambda$; and $x_1=x_3=x$ etc., and further when the B.I.B. designs are such that a treatment in D_2 occurs s times with any other treatment in the same numbered block of D_1 (or D_3) where $x_2=2s$.

4·1. Construction

The design is constructed by generating only b blocks from the b columns of $M_{\alpha\beta\gamma}$.

As an example, we shall consider the construction of the design 4×3^2 in 12 plot blocks. Let the incidence matrices of the three B.I.B. designs used in its construction be

B.I.B. design	D_1 ($v=4$, $b=12$, $k_1=1$, $r_1=3$, $\lambda_1=0$)	D_2 ($v=4$, $b=12$, $k_2=2$, $r_2=4$, $\lambda_2=2$)	D_3 ($v=4$, $b=12$, $k_3=1$, $r_3=3$, $\lambda_3=0$)
Incidence matrix	0 0 0 0 0 0 0 0 0 1 1 1	1 1 1 0 0 0 1 1 1 0 0 0	0 0 0 1 1 1 0 0 0 0 0 0
	0 0 0 0 0 1 0 1 1 0 0 0	1 0 0 1 1 0 1 0 0 1 1 0	0 1 1 0 0 0 0 0 0 0 0 1
	0 0 1 0 1 0 1 0 0 0 0 0	0 1 0 1 0 1 0 1 0 1 0 1	1 0 0 0 0 0 0 0 1 0 1 0
	1 1 0 1 0 0 0 0 0 0 0 0	0 0 1 0 1 1 0 0 1 0 1 1	0 0 0 0 0 0 1 1 0 1 0 0

The design 4×3^2 in 12 plot blocks, each of size 12, thus constructed will be

Level of A	*Blocks from $M_{\alpha\beta\gamma}$*											
	1	2	3	4	5	6	7	8	9	10	11	12
a_0	β	β	β	γ	γ	γ	β	β	β	α	α	α
a_1	β	γ	γ	β	β	α	β	α	α	β	β	γ
a_2	γ	β	α	β	α	β	α	β	γ	β	γ	β
a_3	α	α	β	α	β	β	γ	γ	β	γ	β	β

4·2. Analysis

In this design also only the interactions (BC^2) with 2 d.f. and $A(BC^2)$ with $2(q-1)$ d.f. are affected by block differences. Using the same scheme of constants and notations as earlier, we find that the estimates of the parameters, under the restrictions $\Sigma\, u_i=0$, $\Sigma\, v_i=0$ and $k(t_0+t_2)+k_2t_1=0$, from the normal equations can be obtained independent of the others as

$$\hat{t}_0=P_0/3bk\ ;\ \hat{t}_1=P_1/3bk_2\ ;\ \hat{t}_2=P_2/3bk \qquad ...(13)$$

$$\hat{u}_i=Q_i/\{6r-9(4x-x_2)/3q\} \text{ for } i=0,\ 1,...,\ (q-1) \qquad ...(14)$$

$$\hat{v}_i=R_i/\{6(r+2r_2)-81x_2/3q\} \text{ for } i=0,\ 1,...,\ (q-1) \qquad ...(15)$$

The S.S. due to BC^2, $A(BC^2)_L$ and $A(BC^2)_Q$ can be obtained as $\{(P^2_0+P_2)^2/3bk\}+P^2_1/3bk_2$; $\Sigma\, \hat{u}_iQ_i-(\Sigma\, \hat{u}_i)\,(\Sigma\, Q_i)/q$; and $\Sigma\, \hat{v}_i\, R_i-(\Sigma\, \hat{v}_i)(\Sigma\, R_i)/q$ respectively. The relative loss of information will be

(*i*) $1-(3k/q)$ on the contrast (t_0-t_2) of BC^2

(*ii*) $1-(9kk_2/q^2)$ on the contrast $(t_0-2t_1+t_2)$ of BC^2

(*iii*) $1-(3/2bq)\{2qr-(4x-x_2)\}$ on $A(BC^2)_L$

and (*iv*) $1-(1/2bq)\{2q(r+2r_2)-9x_2\}$ on $A(BC^2)_Q$

which are the same as in the designs detailed in section 3 for the case $k_1=k_3$.

These designs are also non-resolvable and are not equi-replicated. However, when k_i's are chosen such that $k_1=k_2=k_3$ the design would be an equi-replicated one.

Taking the design 4×3^2 constructed in section 4·1 and solving its normal equations, we obtain the following estimates of the affected interaction effects which are independent of the others ;

$$\hat{t}_o=P_o/36\ ;\ \hat{t}_1=P_1/72\ ;\ \hat{t}_2=P_2/36 \qquad ...(13a)$$

$$\hat{u}_i=Q_i/12 \text{ for } i=0,\ 1,\ 2 \text{ and } 3 \qquad ...(14a)$$

$$\hat{v}_i=R_i/63 \text{ for } i=0,\ 1,\ 2 \text{ and } 3 \qquad ...(15a)$$

The S.S. due to BC^2, $A(BC^2)_L$ and $A(BC^2)_Q$ can be obtained as $\{(P^2_0+P^2_2)/36\}+P^2_1/72$; $(1/12)\{\Sigma\, Q^2_i-(\Sigma\, Q_i)^2/4\}$; $(1/63)\{\Sigma\, R^2_i-(\Sigma\, R_i)^2/4\}$ respectively. The relative loss of information is (*i*) 1/4 on $(BC^2)_L$, (*ii*) $-1/8$ on $(BC^2)_Q$, (*iii*) ½ on $A(BC^2)_L$, and (*iv*) 1/8 on $A(BC^2)_Q$. The total relative loss of information will then be $1/4+(-1/8)+(\frac{1}{2}\times3)+(1/8)\times3=2$.

5. Designs $q \times 3^n$ in $q \times 3^p$ Plot Blocks

The methods of construction detailed in sections 2, 3 and 4 for designs $q \times 3^2$ in $3q$ plot blocks can be generalised to the case of designs $q \times 3^n$ in $q \times 3^p$ plot blocks following Sreenath's (1965) technique.

Let A, B_1, B_2,..., B_n be the $(n+1)$ factors at levels q, 3, 3,..., 3 respectively. Firstly, we obtain a $(3^n, 3^{n-p-1})$ confounded symmetrical factorial design D, with B_1, B_2,..., B_n as the n factors, having each of its blocks further divided into 3 sub-blocks of size 3^p by confounding an independent interaction, say, I between the sub-blocks. We define the set of interactions between blocks of D as 'the between block set' and the set of interactions between sub-blocks within blocks as 'the between sub-blocks within blocks set.'

Let α_j, β_j and γ_j denote the sets of 3^p treatment combinations in the j-th block of D, contributing to the totals $(I)_0$, $(I)_1$ and $(I)_2$ respectively. Evidently α_j, β_j and γ_j will be the sub-blocks into which the j-th block of D was divided. We then obtain the three B.I.B. designs D_1, D_2 and D_3 with the levels of the factor A as treatments, as detailed in section 2·1. Now by associating each set of $(\alpha_j, \beta_j, \gamma_j)$ for $j=1, 2,..., 3^{n-p-1}$ with the B.I.B. designs as detailed in sections 2·1, 3·1 and 4·1, the design $q \times 3^n$ in $q \times 3^p$ plot blocks is obtained in b, $2b$. (1/3) and b. (1/3) replications respectively.

In these designs the interactions affected along with the relative loss of information will be :

(*i*) All blonging to the between block set, being completely confounded between blocks. No information is available on them.

(*ii*) All Y, belonging to the 'between sub-blocks within blocks set' each with relative loss of information as that of BC^2 in the corresponding design $q \times 3^2$ in $3q$ plot blocks.

(*iii*) All of the type $A(Y)$ with relative loss of information as that of $A(BC^2)$ in the corresponding design $q \times 3^2$ in $3q$ plot blocks.

The total relative loss of information is $3^{n-p}-1$ (=number of blocks required per replication *minus* one). The S.S. due to affected interactions can be obtained in a manner similar to those in corresponding $q \times 3^2$ designs discussed above.

SUMMARY

To bring about economy in the use of resources, it is desirable to obtain designs involving smallest feasible number of replications providing mutually independent estimates of all the effects. It can further be said that the design need not necessarily be resolvable and equi-replicated. Keeping these points in view, methods of construction alongwith the analysis of the confounded designs $q\times 3^n$ in $q\times 3^p$ plot blocks through balanced incomplete block designs in $b, 2b.(1/3)$ and $b.(1/3)$ replications, where b is the number of blocks in the B.I.B. design used, have been presented in this paper.

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REFERENCES

1. Das, M.N. (1960) : "Fractional replicates as asymmetrical factorial designs" —Jour. Ind. Soc. Agri. Stat., XII, 159-174.
2. Kishen, K. (1960) : "On a class of asymmetrical factorial designs"—Curr. Sci. 29, 465-466.
3. ——— and Srivastava, J.N. (1959) : "Mathematical theory of confounding in asymmetrical and symmetrical factorial designs"—Jour. Ind. Soc. Agri. Stat., XI, 73-110.
4. ——— and Tyagi, B.N. (1961) : "On some methods of construction of asymmetrical factorial experiments"—Curr. Sci., 30, 407-409.
5. Sardana, M.G. and Das, M.N. (1965) : "On the construction and analysis of some confounded asymmetrical factorial designs"—Biometrics, 21, 940-956.
6. ——— and Sreenath, P.R. (1965) : "Construction and analysis of the designs of the type $K^2\times 2^n$ in $K^2\times 2^p$ plot blocks"—Cal. Stat. Assoc. Bull., 14, 74-79.
7. Shah, B.V. (1960) : "On a 5×2^2 factorial design" —Biometrics, 16, 115-118.
8. Sreenath, P.R. (1965) : "On certain methods of construction of confounded asymmetrical factorial designs with smaller number of replications" —Jour. Ind. Soc. Agri. Stat., XVII, 166-181.
9. ——— and Sardana, M.G. (1967) : "Construction and analysis of designs $(2q+1)\times 2^n$ in $(2q+2)\times 2^p$ plot blocks" —Cal. Stat. Assoc. Bull., 16, 29-36.